Generalized fractional integrals on Orlicz spaces (The deepening of function spaces and its environment)

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Dedicated to the memory of Professor Yasuji Takahashi

1 Introduction

This is a joint work with Ryutato Arai and Minglei Shi, and an announcement of [5, 29].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let I_α be the fractional integral operator of order $\alpha \in (0, n)$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.$$

Then it is known as the Hardy-Littlewood-Sobolev theorem that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if $\alpha \in (0, n)$, $p, q \in (1, \infty)$ and $-n/p + \alpha = -n/q$. This boundedness was extended to Orlicz spaces by several authors, see [4, 6, 15, 22, 30, 31, 32], etc. The L^p-L^q boundedness of the commutator $[b, I_\alpha]$ with $b \in \mathrm{BMO}$ was considered by Chanillo [3]. The result was also extended to Orlicz spaces by Fu, Yang and Yuan [7] and Guliyev, Deringoz and Hasanov [8].

In this report we consider generalized fractional integral operators I_ρ on Orlicz spaces. For a function $\rho : (0, \infty) \to (0, \infty)$, the operator I_ρ defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where we always assume that

$$\int_0^1 \frac{\rho(t)}{t} \, dt < \infty. \tag{1.2}$$

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then I_ρ is the usual fractional integral operator I_α. The condition (1.2) is needed for the integral in (1.1) to converge for bounded functions f with compact support.

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The operator I_ρ was introduced in [20] whose partial results were announced in [19]. In these papers we assumed that ρ satisfies the doubling condition;

$$\frac{1}{C_1} \le \frac{\rho(r)}{\rho(s)} \le C_1, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2, \tag{1.3}$$

and that $r \mapsto \rho(r)/r^n$ is almost decreasing;

$$\frac{\rho(s)}{s^n} \le C_2 \frac{\rho(r)}{r^n}, \quad \text{if } r < s, \tag{1.4}$$

where C_1 and C_2 are positive constants independent of $r, s \in (0, \infty)$. Under these conditions we proved the boundedness of I_ρ on Orlicz spaces.

In this report, instead of these conditions, we assume that there exist positive constants C, K_1 and K_2 with $K_1 < K_2$ such that, for all $r > 0$,

$$\sup_{r \le t \le 2r} \rho(t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t}\, dt. \tag{1.5}$$

The condition (1.5) was considered in [25] and also used in [28]. If ρ satisfies (1.3) or (1.4), then ρ satisfies (1.5). Let

$$\rho(r) = \begin{cases} r^n(\log(e/r))^{-1/2}, & 0 < r < 1, \\ e^{-(r-1)}, & 1 \le r < \infty. \end{cases} \tag{1.6}$$

Then ρ satisfies (1.2) and (1.5), but doesn't satisfy (1.3) or (1.4). Therefore, the results in this report are improvement of one in [20]. Moreover, we consider the commutator $[b, I_\rho]$ with functions b in generalized Campanato spaces. To prove the boundedness of $[b, I_\rho]$ on Orlicz spaces we need the sharp maximal operator $M^\sharp$ and generalized fractional maximal operators M_ρ, see (1.8) and (1.9) below for their definitions. Moreover, we need a generalization of the Young function.

First we recall the definition of the generalized Campanato space and the sharp maximal and generalized fractional maximal operators. We denote by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius r, that is,

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a ball B, let

$$f_B = \fint_B f = \fint_B f(y)\, dy = \frac{1}{|B|} \int_B f(y)\, dy. \tag{1.7}$$

Definition 1.1. For $p \in [1, \infty)$ and $\psi : (0, \infty) \to (0, \infty)$, let $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ be the sets of all functions f such that the following functional is finite:

$$\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)} = \sup_{B = B(x,r)} \frac{1}{\psi(r)} \left(\fint_B |f(y) - f_B|^p\, dy \right)^{1/p},$$

where the supremam is taken over all balls $B(x, r)$ in $\mathbb{R}^n$.

Then $\|f\|_{\mathcal{L}_{p,\psi}(\mathbb{R}^n)}$ is a norm modulo constant functions and thereby $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ is a Banach space. If $p=1$ and $\psi \equiv 1$, then $\mathcal{L}_{p,\psi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$.

The sharp maximal operator $M^\sharp$ is defined by

$$M^\sharp f(x) = \sup_{B \ni x} \fint_B |f(y) - f_B|\, dy, \quad x \in \mathbb{R}^n, \tag{1.8}$$

where the supremum is taken over all balls B containing x. For a function $\rho : (0,\infty) \to (0,\infty)$, let

$$M_\rho f(x) = \sup_{B(z,r) \ni x} \rho(r) \fint_{B(z,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n, \tag{1.9}$$

where the supremum is taken over all balls B containing x. We don't assume the condition (1.2) or (1.5) on the definition of M_ρ. The operator M_ρ was studied in [27] on generalized Morrey spaces. If $\rho(B) = |B|^{\alpha/n}$, then M_ρ is the usual fractional maximal operator M_α. If $\rho \equiv 1$, then M_ρ is the Hardy-Littlewood maximal operator M, that is,

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

The operator M is bounded from $L^p(\mathbb{R}^n)$ to itself, if $1 < p \le \infty$.

It is known that the usual fractional maximal operator M_α is dominated pointwise by the fractional integral operator I_α, that is, $M_\alpha f(x) \le C I_\alpha |f|(x)$ for all $x \in \mathbb{R}^n$. Then the boundedness of M_α follows from one of I_α. However, we need a better estimate on M_ρ than I_ρ to prove the boundedness of the commutator $[b, I_\rho]$. In this report we give a necessary and sufficient condition of the boundedness of M_ρ.

Here we recall the proof of Hardy-Littlewood-Sobolev theorem by Hedberg [11].

Theorem 1.1 (Hardy-Littlewood-Sobolev (1928, 1932, 1938)).
If $\alpha \in (0,n)$, $p,q \in (1,\infty)$ and $-n/p + \alpha = -n/q$, then

$$I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \quad \textit{bounded.}$$

Proof by Hedberg (1972). We prove that, for $f \in L^p(\mathbb{R}^n)$ with $\|f\|_{L^p} = 1$,

$$|I_\alpha f(x)|^q \lesssim Mf(x)^p, \quad x \in \mathbb{R}^n.$$

Then, using the boundedness of the Hardy-Littlewood maximal operator M on $L^p(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |I_\alpha f|^q \lesssim \int_{\mathbb{R}^n} (Mf)^p \lesssim \int_{\mathbb{R}^n} |f|^p = 1.$$

To prove the above pointwise estimate, let

$$|I_\alpha f(x)| \le \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy = \int_{|x-y|<r} + \int_{|x-y|\ge r} = J_1 + J_2.$$

Then we can get

$$J_1 \le Mf(x) \int_{|x|<r} \frac{1}{|x|^{n-\alpha}} \lesssim Mf(x)\, r^{\alpha},$$

$$J_2 \le \|f\|_{L^p} \left(\int_{|x|\ge r} \left(\frac{1}{|x|^{n-\alpha}} \right)^{p'} dy \right)^{1/p'} \sim r^{-n/q}.$$

Let $r = Mf(x)^{-p/n}$. Then $r^{\alpha} = Mf(x)^{-\alpha p/n} = Mf(x)^{p/q-1}$ and

$$|I_\alpha f(x)| \le J_1 + J_2 \lesssim Mf(x)^{p/q}. \qquad \square$$

In this report, to prove the boundedness of I_ρ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, we show the pointwise estimate

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1}\right) \le \Phi\left(\frac{Mf(x)}{C_0}\right), \quad x \in \mathbb{R}^n,$$

for $f \in L^\Phi(\mathbb{R}^n)$ with $\|f\|_{L^\Phi} = 1$.

2 Young functions and Orlicz spaces

For an increasing function $\Phi : [0,\infty] \to [0,\infty]$, let

$$a(\Phi) = \sup\{t \ge 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \ge 0 : \Phi(t) = \infty\}.$$

Then $0 \le a(\Phi) \le b(\Phi) \le \infty$. Let $\bar{\varPhi}$ be the set of all increasing functions $\Phi : [0,\infty] \to [0,\infty]$ such that

$$\lim_{t\to+0} \Phi(t) = \Phi(0) = 0, \tag{2.1}$$

$$\Phi \text{ is left continuous on } [0, b(\Phi)), \tag{2.2}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty, \tag{2.3}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t\to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty). \tag{2.4}$$

Any function in $\bar{\varPhi}$ is neither identically zero nor identically infinity on $(0,\infty)$.

For $\Phi \in \bar{\varPhi}$, we recall the generalized inverse of Φ in the sense of O'Neil [22, Definition 1.2]. For $\Phi \in \bar{\varPhi}$ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \ge 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases} \tag{2.5}$$

Then Φ^{-1} is finite and right continuous on $[0,\infty)$ and positive on $(0,\infty)$. If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, we have the following relation, which is a generalization of Property 1.3 in [22].

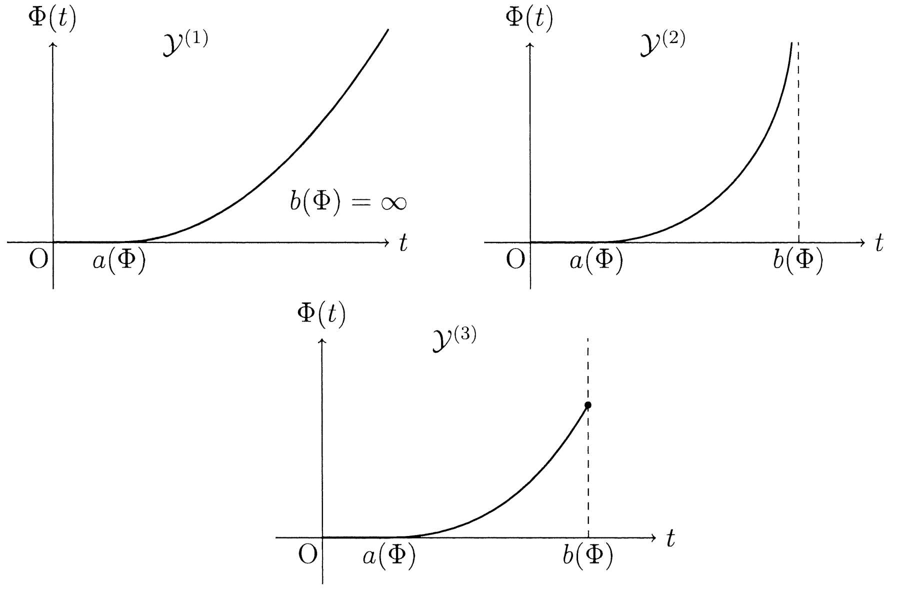
$$\Phi(\Phi^{-1}(u)) \le u \le \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0,\infty]. \tag{2.6}$$

Definition 2.1. A function $\Phi \in \bar{\varPhi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$.

By the convexity, any Young function Φ is continuous on $[0, b(\Phi))$ and strictly increasing on $[a(\Phi), b(\Phi)]$.

We define three subsets $\mathcal{Y}^{(i)}$ $(i = 1, 2, 3)$ of Young functions as

$$\mathcal{Y}^{(1)} = \{\Phi \in \mathit{\Phi}_Y : b(\Phi) = \infty\},$$
$$\mathcal{Y}^{(2)} = \{\Phi \in \mathit{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) = \infty\},$$
$$\mathcal{Y}^{(3)} = \{\Phi \in \mathit{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) < \infty\}.$$



For $\Phi, \Psi \in \bar{\mathit{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

Definition 2.2. (i) Let $\mathit{\Phi}_Y$ be the set of all Young functions.

(ii) Let $\bar{\mathit{\Phi}}_Y$ be the set of all $\Phi \in \bar{\mathit{\Phi}}$ such that $\Phi \approx \Psi$ for some $\Psi \in \mathit{\Phi}_Y$.

For $\Phi \in \bar{\mathit{\Phi}}_Y$, we define the Orlicz space $L^\Phi(\mathbb{R}^n)$ and the weak Orlicz space $\mathrm{w}L^\Phi(\mathbb{R}^n)$. Let $L^0(\mathbb{R}^n)$ be the set of all complex valued measurable functions on $\mathbb{R}^n$.

Definition 2.3. For a function $\Phi \in \bar{\mathit{\Phi}}_Y$, let

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^0(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)\, dx < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\},$$

$$\mathrm{w}L^\Phi(\Omega) = \left\{ f \in L^0(\mathbb{R}^n) : \sup_{t \in (0,\infty)} \Phi(t)\, m(\epsilon f, t) < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{\mathrm{w}L^\Phi} = \inf \left\{ \lambda > 0 : \sup_{t \in (0,\infty)} \Phi(t)\, m\left(\frac{f}{\lambda}, t\right) \le 1 \right\},$$

$$\text{where} \quad m(f, t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|.$$

Then $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{\mathrm{w}L^\Phi}$ are quasi-norms and $L^\Phi(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $\Phi \in \mathit{\Phi}_Y$, then $\|\cdot\|_{L^\Phi}$ is a norm and thereby $L^\Phi(\mathbb{R}^n)$ is a Banach space. For $\Phi, \Psi \in \bar{\mathit{\Phi}}_Y$, if $\Phi \approx \Psi$, then $L^\Phi(\mathbb{R}^n) = L^\Psi(\mathbb{R}^n)$ and quasi-norms $\|\cdot\|_{L^\Phi}$ and $\|\cdot\|_{L^\Psi}$ are equivalent. Orlicz spaces are introduced by [23, 24]. For the theory of Orlicz spaces, see [14, 15, 16, 17, 26] for example.

Definition 2.4. (i) A function $\Phi \in \bar{\mathit{\Phi}}$ is said to satisfy the Δ_2-condition, denote $\Phi \in \bar{\Delta}_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t) \quad \text{for all } t > 0. \tag{2.7}$$

(ii) A function $\Phi \in \bar{\mathit{\Phi}}$ is said to satisfy the ∇_2-condition, denote $\Phi \in \bar{\nabla}_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \le \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0. \tag{2.8}$$

(iii) Let $\Delta_2 = \mathit{\Phi}_Y \cap \bar{\Delta}_2$ and $\nabla_2 = \mathit{\Phi}_Y \cap \bar{\nabla}_2$.

The following theorem is known, see [15, Theorem 1.2.1] for example.

Theorem 2.1. *Let $\Phi \in \bar{\mathit{\Phi}}_Y$. Then M is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Phi(\mathbb{R}^n)$. Moreover, if $\Phi \in \bar{\nabla}_2$, then M is bounded on $L^\Phi(\mathbb{R}^n)$.*

See also [4, 12, 13] for the Hardy-Littlewood maximal operator on Orlicz spaces.

3 Results

Theorem 3.1. *Let $\rho : (0,\infty) \to (0,\infty)$ satisfy (1.2) and (1.5), and let $\Phi, \Psi \in \bar{\mathit{\Phi}}_Y$, $a(\Phi) = 0$ and $b(\Phi) = \infty$. Assume that there exists a positive constant A such that, for all $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\,dt\ \Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\,dt \le A\Psi^{-1}(1/r^n). \tag{3.1}$$

Then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$\Psi\left(\frac{|I_\rho f(x)|}{C_1\|f\|_{L^\Phi}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right). \tag{3.2}$$

Consequently, I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \bar{\nabla}_2$, then I_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

See [5, 21] for examples of $\Phi, \Psi \in \bar{\mathit{\Phi}}_Y$ which satisfy the assumption in Theorem 3.1. See also [18] for the boundedness of I_ρ on Orlicz space $L^\Phi(\Omega)$ with bounded domain $\Omega \subset \mathbb{R}^n$.

Next we state the result on the operator M_ρ defined by (1.9) in which we don't assume (1.2) or (1.5).

Theorem 3.2. *Let $\rho : (0, \infty) \to (0, \infty)$, and let $\Phi, \Psi \in \bar{\Phi}_Y$.*

(i) *Assume that there exists a positive constant A such that, for all $r \in (0, \infty)$,*

$$\left(\sup_{0<t\le r} \rho(t) \right) \Phi^{-1}(1/r^n) \le A\Psi^{-1}(1/r^n). \tag{3.3}$$

Then, for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in L^\Phi(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$\Psi\left(\frac{|M_\rho f(x)|}{C_1\|f\|_{L^\Phi}} \right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}} \right). \tag{3.4}$$

Consequently, M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \bar{\nabla}_2$, then M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

(ii) *Conversely, if M_ρ is bounded from $L^\Phi(\mathbb{R}^n)$ to $\mathrm{w}L^\Psi(\mathbb{R}^n)$, then (3.3) holds for some A and all $r \in (0, \infty)$.*

Theorem 3.3. *Let $\rho : (0, \infty) \to (0, \infty)$ satisfy (1.2).*

(i) *Let $\Phi, \Psi \in \bar{\Delta}_2 \cap \bar{\nabla}_2$. Assume that $r \mapsto \rho(r)/r^{n-\epsilon}$ is almost decreasing for some $\epsilon \in (0, n)$. Assume also that there exists a positive constant A and $\Theta \in \bar{\nabla}_2$ such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\, dt\ \Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)\,\Phi^{-1}(1/t^n)}{t}\, dt \le A\Theta^{-1}(1/r^n), \tag{3.5}$$

$$\psi(r)\Theta^{-1}(1/r^n) \le A\Psi^{-1}(1/r^n), \tag{3.6}$$

and that there exist a positive constant C_ρ such that, for all $r, s \in (0, \infty)$,

$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \le C_\rho\, |r - s| \frac{1}{r^{n+1}} \int_0^r \frac{\rho(t)}{t}\, dt, \quad \text{if } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{3.7}$$

If $b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n)$, then $[b, I_\rho]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and there exists a positive constant C such that, for all $f \in L^\Phi(\mathbb{R}^n)$,

$$\|[b, I_\rho]f\|_{L^\Psi} \le C\|b\|_{\mathcal{L}_{1,\psi}}\|f\|_{L^\Phi}. \tag{3.8}$$

(ii) *Conversely, let $\Phi, \Psi \in \bar{\Phi}_Y$, and assume that there exists a positive constant A such that, for all $r \in (0, \infty)$,*

$$\Psi^{-1}(1/r^n) \le Ar^\alpha\psi(r)\Phi^{-1}(1/r^n).$$

If $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, then b is in $\mathcal{L}_{1,\psi}(\mathbb{R}^n)$ and there exists a positive constant C, independent of b, such that

$$\|b\|_{\mathcal{L}_{1,\psi}} \le C\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi}, \tag{3.9}$$

where $\|[b, I_\alpha]\|_{L^\Phi \to L^\Psi}$ is the operator norm of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

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